Stroppel

Khovanov homology from rep. theory

Want : To categorify tensor products of finite dim modules

$U_q(\mathfrak{sl}_2)$-modules

$\rightarrow$ RT-tangle inv.

$\rightarrow$ CK-invariants ?

Recall $\mathfrak{g}$: a ss cpx Lie alg $\rightsquigarrow \mathcal{O}(\mathfrak{g})$

$\lambda \in \mathfrak{h}^*/W \rightsquigarrow \mathcal{O}(\mathfrak{g})_\lambda \subset \mathcal{O}(\mathfrak{g})$ W = Weyl group

From now on $\mathfrak{g} = \mathfrak{sl}_n$

categorify $\underbrace{V \otimes \cdots \otimes V}_{n}$ $V = \mathbb{C}^2 \circlearrowleft U_q(\mathfrak{sl}_2)$

~~pick up~~ for each $0 \le i \le n$ pick $\lambda_i \in \mathfrak{h}^*$ integral dominant

s.t. stabilizer of λ_i is $S_i \times S_{n-i} < S_n = W$

e.g. $\mathfrak{sl}_3$

$S_0 \times S_3$ $S_1 \times S_2$ $S_2 \times S_1$ $S_3 \times S_0$

orbit $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$ elements

Th There is an isom. $K_0\left(\bigoplus_{i=0}^{n} \mathcal{O}(\mathfrak{sl}_n)_{\lambda_i}\right) \cong V^{\otimes n}$ s.t.

graded ---- deep

$\mathbb{C}[q,q^{-1}]$-iso.

Verma module $\longleftrightarrow$ std basis

$\triangle$ Simple module $\longleftrightarrow$ dual canonical basis

twisted indec. proj. modules $\longleftrightarrow$ canonical basis

(tilting)

action of E, F, K $\longleftrightarrow$ tensoring with natural rep $\mathbb{C}^n$, its dual

grading shift

Remarks

can be generalized to arbitrary tensor products of finite dimensional modules of $U_q(sl_2)$)

using categories of Harish-Chandra modules (jt with Frenkel-Khovanov)

generalization of ① works well although the categories are not highest wt)

jt work with Mazorchuk

Picture so far convenient for $U_q(sl_2)$-action

now want TL-action

$\mathfrak{g} = sl_n \quad 0 \leq i \leq n \quad$ fix $\quad \mathfrak{p}_i = \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \begin{array}{l} \}\, i \\ \}\, n-i \end{array}$

$\mathcal{O}^{\mathfrak{p}_i}(\mathfrak{g})_0 \subset \mathcal{O}(\mathfrak{g})_0$

all objects which are loc. finite w.r.t. $\mathcal{U}(\mathfrak{p}_i)$)

simple objects $L(x) \quad x \in S_i \times S_{n-i} \backslash S_n$

shortest length coset representative

Koszul duality で上の picture につづく

graded version

Th. 1) $K_0\left(\bigoplus_{i=0}^{n} \mathcal{O}^{p_i}(\mathfrak{sl}_n)_0\text{-mod.}\right) \cong V^{\otimes n}$ parabolic Vermas $\leftrightarrow$ standard basis

2) $\left\{\begin{array}{l}\text{Grothendieck group}\\ \text{of proj functors}\end{array}\right\} \cong$ Temperley-Lieb $T_{n,q}$

in the graded version

restriction of proj. functors on $\mathcal{O}$

3) This extends to an invariant of tangles + cobordism as follows

{tangles}		Cat	
objects	$\mathbb{N}$	$\mathcal{D}^b\left(\bigoplus_{i=0}^{n} \mathcal{O}^{p_i}(\mathfrak{sl}_n)_0\right)$	$n \in \mathbb{N}$
morphism	tangles	functors	
2-morphism	cobordism	natural transf. / upto scalars	

4) On the Grothendieck group RT

connection to $D^b(\mathrm{Coh}\, Y_n)$?

Conj. $D^b(\mathrm{Coh}\, Y_n) \cong D^b(B\text{-grmod}) \underset{\text{Koszul dual}}{\cong} D^b\left(\bigoplus_{i=0}^{n} \mathcal{O}^{p_i}(\mathfrak{sl}_n)_0\right)$

NB. Temp.Lieb. ~~is~~ are exact functors in the $K_0(\oplus \mathcal{O}^{p_i}(\mathfrak{sl}_n)\text{-mod})$ side

General fact

$\mathcal{O}^{\mathfrak{p}_i}(\mathfrak{sl}_n)_0 \cong \operatorname{mod} A_n^i$ for some f.d. algebra A_n^i

Braden gave description via generators & relations
(LHS $\cong$ perv. sheaves on Grassmann)
not obvious that the algebra is graded
he conjectured H_n is a subquotient algebra of A_{2n}^n

Th H_n is a natural subalgebra of A_{2n}^n, namely

$$H_n = \operatorname{End}_{A_{2n}^n}(\oplus \text{ indec. proj. \& injective})$$

Serre functor is trivial!

restricting functors from thm to LHS gives exactly Khovanov's picture.

A_n^i is better than H_n is not Koszul.

But the trade off is not computable.

A_n^i via diagrams
write a basis of $V^{\otimes n}$ as "spin chains" using $\wedge$, $\vee$
e.g. $V \otimes V$ $\quad \wedge\wedge \mid \wedge\vee \quad \vee\wedge \mid \vee\vee$

ith weight sp. = # downs is i

parabolic Vermas $\leftrightarrow$ spin chains

Rule I) given a spin chain S connect all $\vee\wedge$ via a cup diagram (if they are nbd)

e.g. $\wedge\vee \to$ do nothing $\uparrow\downarrow$

$\vee\wedge \to$ 

result cup diagram $C(S)$ with probably also $\uparrow\downarrow$

Lemma S a spin chain, $M(S)$ corresponding Verma, $P(S)$ its proj. cover

$$\underbrace{[P(S) : M(t)\langle j\rangle]}_{\text{grading shift}} = \#\text{ orientations of } C(S) \text{ of degree } j \text{ and type } t$$

e.g.  or  degree = # of clockwise cups

type $\wedge\vee$ $\vee\wedge$

$P(\vee\wedge) = M(\vee\wedge)$
$M(\wedge\vee)\langle 1\rangle$

$\uparrow\downarrow$ only one orientation $P(\wedge\vee) = M(\wedge\vee)$

Rule II) Do the same thing with caps instead of cups

gives $[M(S) ; L(t)\langle j\rangle]$

interpretation of BGG-reciprocity

$$[P(S) : M(t)\langle j\rangle] = [M(t) : L(S)\langle j\rangle]$$

$$\dim \operatorname{Hom}(P(S), P(z)) = \sum_{y,k,j} [P(S), M(y)\langle k\rangle] \, [M(y), L(z)\langle j\rangle]$$

Th 1) The algebra A_n^i has a graded vector spae basis
given by all possible oriented cup-cap diagram.

e.g. A_2^1 

grading clockwise cup/cap shift by 1

0 1 1 2 0

2) multiplication just as in Khovanov's

3) The cup-cap diagrams which gives closed circles only
form a subalgebra which is H_n in the case A_{2n}^n.

Rem.